

Embedding variables in the canonical theory of gravitating shells

Petr Hájíček

Institut für Theoretische Physik,
Universität Bern,
Sidlerstrasse 5, 3012 Bern, Switzerland,
E-mail: hajicek@itp.unibe.ch,
Telephon: (41)(31) 631 86 25, FAX: 631 38 21.

Claus Kiefer

Fakultät für Physik,
Universität Freiburg,
Hermann-Herder-Strasse 3, 79104 Freiburg, Germany.

Abstract

A thin shell of light-like dust with its own gravitational field is studied in the special case of spherical symmetry. The action functional for this system due to Louko, Whiting, and Friedman is reduced to Kuchař form: the new variables are embeddings, their conjugate momenta, and Dirac observables. The concepts of background manifold and covariant gauge fixing, that underlie these variables, are reformulated in a way that implies the uniqueness and gauge invariance of the background manifold. The reduced dynamics describes motion on this background manifold.

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1 Introduction

The phenomenon of gravitational collapse leads to serious problems in the classical theory of gravity. The structure of the resulting singularities contradict the foundations of the theory such as the equivalence principle. Thus, the existence of singularity theorems [1] may constitute a strong motivation to address the quantisation of the gravitational field, in the hope that such a framework can avoid the occurrence of singularities.

A prominent feature of the classical collapse is the existence of horizons which appear sooner than the singularity. Such horizons not only imply that the singularity is inevitable (which is, roughly, the content of the singularity theorems), they also seem to prevent any object or information from leaving the region of collapse and from coming back to the asymptotic region. It is the existence of horizons that makes gravity so different and the problem of collapse so difficult. On the other hand, the problem of gravitational collapse is a very special one. For its solution, a complete quantum theory of gravitation may be as little needed as the complete quantum electrodynamics was needed for the first calculations of atomic spectra.

Motivated by these ideas, we consider the quantum theory of a spherically symmetric thin shell and its gravitational field. This is, in fact, a quite popular system. For example, it was used to study the motion of domain walls in the early Universe [2], of black-hole evaporation [3], of quantum black holes [4], and many others. In [5], [6] and [7], gravitational collapse of such a thin shell in its own gravitational field has been studied. The result was analogous to what is known about the s -mode of the Coulomb problem. Two aspects of this result were surprising. First, for low-mass shells, there were stationary states with Sommerfeld spectrum and scattering states with the wave packets describing the shell bouncing and re-expanding. The evolution was unitary. Second, there was an analogue to the critical charge in the relativistic quantum mechanics of atoms. The role of charge was played by the rest mass of the shell (not to be confused with its total energy) and the critical value of the rest mass was about one Planck mass. As in the case of relativistic atoms, the quantum-mechanical description breaks down for supercritical “charges”.

To understand these results was difficult. Even the simple scattering of the subcritical shells admitted several different interpretations. There were two problems. First, the radial coordinate of the shell, which served as the argument of the wave function, did not possess the status of a quantum observable. The Coulomb-like potential prevented one from constructing a position operator similar to the Newton-Wigner operator. Second, the model was completely reduced to the physical degrees of freedom, which in this case is just the radius of the shell. However, the value of the radius is not as informative in a black-hole spacetime as it is, for example, in Minkowski spacetime: points with the same value of radial coordinate can lie in

different asymptotically flat regions. These regions are separated by horizons. The tools that were at one's disposal in [5], [6] and [7] did not allow to decide whether the shell created a horizon and then, consequently, re-expanded behind this horizon into a different asymptotically flat section, or whether it did not create any horizon and re-expanded into the same region from which it collapsed.

In [8], two remedies have been proposed. The first is to work with a null (light-like) shell. The classical dynamics of such a shell is equivalent to that of free photons on flat two-dimensional spacetime (the “charge” is zero). For such a system, there is a well-defined position operator [9]. Moreover, it admits a simple description of its asymptotic states, unlike Coulomb scattering.

The second idea is that the equation of motion for $r(t)$, which has been obtained from Einstein's equations, must in fact result from a reduction to true degrees of freedom of an action that contains the shell as well as the gravitational field. Indeed, in the spherically-symmetric case, the gravitational degrees of freedom consist only of the gauge and the dependent ones. It seems that the reduction has been performed in such a way that the information about the geometry of spacetime has been lost. We shall, therefore, perform the reduction explicitly in a careful way. This is the main purpose of this paper.

In general, the reduction procedure consists of two steps: the choice of gauge and the solution of constraints. There exists a particular form of the gravitational action that is effectively reduced, but which still contains some information about the geometry of spacetime: the so-called *Kuchař decomposition* [10], [11]. Kuchař variables are neatly separated into pure gauge ones (so-called embeddings), dependent ones that are conjugate to the embeddings, and physical degrees of freedom. Some progress in understanding Kuchař decomposition has been achieved recently [12], [13]: general existence of the decomposition has been shown, and the crucial role of gauge choice in it has been recognised. The nature of gauge choice in quantum gravity has also been elucidated. Two important notions have been introduced: *background manifold* and *covariant gauge fixing*. As a matter of fact, the present paper is the first practical application of these concepts. It deals with the *classical* canonical analysis of the null-dust shell. Its main result is the *explicit construction* of the Kuchař decomposition. This then serves as the starting point for the quantum theory of the shell, which will be presented in a separate paper.

The plan of the paper is as follows. Sect. 2 explains the notions of background manifold and covariant gauge fixing in a new and clear form. This enables us to show the gauge invariance, the uniqueness and some additional structures of the background manifold, which will be necessary for the interpretation of the shell quantum mechanics. Some important points of [13] are then summarised in the new language. Sect. 3 describes the solutions of Einstein's equations containing the

shell. After fixing the gauge, these solutions are reformulated as a set of parameter-dependent metric fields and shell trajectories on the background manifold. The parameters distinguish the physically different solutions and will play the role of the physical variables in the Kuchař decomposition. The representative metric fields and shell trajectories will be used to define the transformation from the ADM to Kuchař variables on the constraint hyper-surface. This transformation is performed in Sect. 4. Starting point is a (non-reduced) Hamiltonian action principle [14] for the spherically-symmetric shell and its gravitational field. In Sect. 4, an extension of the results to a whole neighbourhood of the constraint hyper-surface is performed using the methods and theorems of [13]. The final action, the variables in it and some discussion can be found in Sect. 5.

2 Background manifold and covariant gauge fixing

In this section we introduce the two basic notions of background manifold and covariant gauge fixing, restricting ourselves, for the sake of simplicity, to vacuum general relativity. The language is slightly different from that used in [13] so that we can prove more results; we also summarise some points that are important for the paper.

Let \mathcal{M} be a four-manifold that admits a Lorentzian metric field g such that (\mathcal{M}, g) is a globally hyperbolic spacetime. Dynamically maximal solutions of Einstein's equations are always of this form [15]. Then according to a theorem of Geroch [16], $\mathcal{M} = \Sigma \times \mathbf{R}$, where Σ is an initial data manifold. Topological sectors of general relativity are uniquely associated with different three-manifolds Σ . \mathcal{M} is called *background manifold*. In this way, each topological sector determines a *unique* background manifold \mathcal{M} .

All diffeomorphisms of \mathcal{M} form the group $\text{Diff}\mathcal{M}$; this is considered as the “gauge group” of general relativity. Observe that the group depends on the topological sector chosen, i.e., on \mathcal{M} , and that the manifold structure of \mathcal{M} itself is *gauge invariant*. Single points of \mathcal{M} are, however, not gauge invariant, since they are being pushed around by $\text{Diff}\mathcal{M}$. In some important cases, not $\text{Diff}\mathcal{M}$ but some of its subgroups play the role of gauge group. For example, in the case of asymptotically flat space-times, only those diffeomorphisms are considered as gauge transformations that become sufficiently quickly trivial at infinity. In general, in such cases, \mathcal{M} is equipped with some gauge-invariant structure in addition to the naked manifold one.

Let $\varphi \in \text{Diff}\mathcal{M}$ and let g be a Lorentzian metric on \mathcal{M} . Then the inverse pull-back associated with φ maps g into another metric g' , $g' = (\varphi^{-1})_*g$. In this way,

the group $\text{Diff}\mathcal{M}$ acts on the space $\text{Riem}\mathcal{M}$ of all (suitably restricted) Lorentzian metrics on \mathcal{M} . The action is not transitive and so there are non-trivial orbits of the group in $\text{Riem}\mathcal{M}$. Such orbits are called *geometries* and the quotient space $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M}$ is the space of geometries. Let $\pi : \text{Riem}\mathcal{M} \mapsto \text{Riem}\mathcal{M}/\text{Diff}\mathcal{M}$ be the natural projection for the quotient. One can equip the space of geometries with some additional structure, e.g., a topology, starting from a structure of $\text{Riem}\mathcal{M}$ and using the projection.

Suppose that we manage, at least for some open set $U \subset \text{Riem}\mathcal{M}/\text{Diff}\mathcal{M}$, to specify a section σ . This is a map,

$$\sigma : \text{Riem}\mathcal{M}/\text{Diff}\mathcal{M} \mapsto \text{Riem}\mathcal{M}$$

such that $\pi \circ \sigma = \text{id}$. The meaning of such a section is that a particular *representative metric* on \mathcal{M} is chosen for each geometry in U . This is exactly what has been called *covariant gauge fixing* in [13]. Clearly, the transformation between two covariant gauge fixings is not a single diffeomorphism, but an element of the Bergmann-Komar group [17].

Given a covariant gauge fixing σ on U , one can use it to construct a map from $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M} \times \text{Emb}(\Sigma, \mathcal{M})$, where $\text{Emb}(\Sigma, \mathcal{M})$ is the space of embeddings of the initial data surface Σ into \mathcal{M} , to the ADM phase space of general relativity. The construction has been described in [13] and it goes, roughly, as follows. Let $\gamma \in U$ and let $\sigma(\gamma)$ be the representative metric on \mathcal{M} . Let $X : \Sigma \mapsto \mathcal{M}$ be an embedding. Then $\sigma(\gamma)$ determines the first, $q_{kl}(x)$, and the second, $K_{kl}(x)$, fundamental forms of the surface $X(\Sigma)$ in the spacetime $(\mathcal{M}, \sigma(\gamma))$. The corresponding point $(q_{kl}(x), \pi^{kl}(x))$ of the ADM phase space can be obtained in the well known way from $q_{kl}(x)$ and $K_{kl}(x)$.

In [13], this transformation has been restricted to give only the points of the constraint surface Γ of the ADM phase space; moreover, only those points of Γ have been selected, where the evolved space-times do not admit any isometry. Then the map from $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M} \times \text{Emb}(\Sigma, \mathcal{M})$ to Γ has been shown to be invertible and extensible to a neighbourhood of $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M} \times \text{Emb}(\Sigma, \mathcal{M})$ in the larger space $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M} \times T^*\text{Emb}(\Sigma, \mathcal{M})$, which has then been mapped to a neighbourhood of Γ in the ADM phase space. Next, the Darboux-Weinstein theorem has been employed to prove some nice symplectic properties of the map. These properties then make the map to a general transformation of the ADM to the Kuchař variables.

This procedure will here be applied to the model of spherically-symmetric thin gravitating shell in the subsequent sections. We shall find in the next section the set of representative solutions for Einstein's equations for each physically distinct situation of the shell because, as has been shown in [13], this part of the section σ suffices completely to construct the above map to the constraint surface of our model.

3 Einstein dynamics of the shell

Any spherically-symmetric solution of Einstein's equations with a thin null shell as the source has a simple structure. Inside the shell, the spacetime is flat; outside the shell, it is isometric to a part of the Schwarzschild spacetime of mass M . The two geometries must be stuck together along a spherically-symmetric null hyper-surface so that the points with the same values of the radial coordinate R coincide.

All physically distinct solutions can be labeled by three parameters: $\eta \in \{-1, +1\}$, distinguishing between the outgoing ($\eta = +1$) and in-going ($\eta = -1$) null surfaces; the asymptotic time of the surface, i.e., the retarded time $u = T - R \in (-\infty, \infty)$ for $\eta = +1$, and the advanced time $v = T + R \in (-\infty, \infty)$ for $\eta = -1$; and the mass $M \in (0, \infty)$. An in-going shell creates a black-hole (event) horizon at $R = 2M$ and ends up in the singularity at $R = 0$. The outgoing shell starts from the singularity at $R = 0$ and emerges from a white-hole (particle) horizon at $R = 2M$.

We can write down the metric in the case $\eta = 1$ with the help of retarded Eddington-Finkelstein coordinates \tilde{U} , R , ϑ and φ . $\tilde{U} = u$ is the trajectory of the shell, $\tilde{U} > u$ is a part of Minkowski spacetime,

$$ds^2 = -d\tilde{U}^2 - 2d\tilde{U}dR + R^2d\Omega^2, \quad (1)$$

and $\tilde{U} < u$ is a part of Schwarzschild spacetime,

$$ds^2 = -\left(1 - \frac{2M}{R}\right)d\tilde{U}^2 - 2d\tilde{U}dR + R^2d\Omega^2. \quad (2)$$

Similarly, for $\eta = -1$, the advanced Eddington-Finkelstein coordinates are \tilde{V} , R , ϑ and φ , and $\tilde{V} = v$ is the shell. Inside the shell, $\tilde{V} < v$,

$$ds^2 = -d\tilde{V}^2 + 2d\tilde{V}dR + R^2d\Omega^2, \quad (3)$$

and outside the shell, $\tilde{V} > v$,

$$ds^2 = -\left(1 - \frac{2M}{R}\right)d\tilde{V}^2 + 2d\tilde{V}dR + R^2d\Omega^2. \quad (4)$$

Let us denote the spacetime given by the triple of parameters η , M and w by (η, M, w) , where $w = u$ for $\eta = 1$ and $w = v$ for $\eta = -1$.

We observe that the two space-times (η, M, w_1) and (η, M, w_2) are isometric, the isometry sending the point $(\tilde{U}, R, \vartheta, \varphi)$ into $(\tilde{U} + w_2 - w_1, R, \vartheta, \varphi)$. Hence, the geometries of the solutions that differ only in the value of the parameter w are equal. Yet, the physical situations they represent are different; this is similar to the motion of a free mass point in Minkowski spacetime. For each two different trajectories, there is a Poincaré transformation that sends the first into the second. Still, the two

motions are physically different because they look differently from one fixed inertial frame. For the shell, instead of an inertial frame, we imagine that there is a *fixed* asymptotic family of observers. The group of these isometries is a symmetry group rather than a gauge group. It can (and will) be employed to define a time evolution.

Another interesting isometry is the map $\mathcal{T} : (\eta, M, w_1) \mapsto (-\eta, M, w_2)$ defined for $\eta = +1$ and arbitrary w_1 and w_2 by the Eddington-Finkelstein coordinates as follows:

$$\mathcal{T}(\tilde{U}_1, R_1, \vartheta_1, \varphi_1) = (\tilde{V}_2, R_2, \vartheta_2, \varphi_2) ,$$

where

$$\tilde{V}_2 = -\tilde{U}_1 + w_1 + w_2, \quad R_2 = R_1, \quad \vartheta_2 = \vartheta_1, \quad \varphi_2 = \varphi_1 .$$

For $\eta = -1$, we just take the inverse of the above so that $\mathcal{T}^2 = \text{id}$. \mathcal{T} can be viewed as a time reversal symmetry.

The Eddington-Finkelstein coordinates may be nicely adapted to the symmetry and may simplify the metric, but they do not define a covariant gauge fixing. Indeed, the identification of the points $(\tilde{U}_1, R_1, \vartheta_1, \varphi_1)$ of the solution $(+1, M_1, w_1)$ with the points $(\tilde{V}_2, R_2, \vartheta_2, \varphi_2)$ of $(-1, M_2, w_2)$ satisfying the relations $\tilde{V}_2 = \tilde{U}_1$, $R_2 = R_1$, $\vartheta_2 = \vartheta_1$ and $\varphi_2 = \varphi_1$ will invert the time orientation of the asymptotic observers, which is to stay gauge invariant. We need, however, a covariant gauge fixing if we are to transform the action to the Kuchař form. The rest of this section will be devoted to a choice of gauge that will be convenient for this problem.

To start with, we have to specify the background manifold. Our model comprises only the spherically-symmetric part of general relativity with the shell. We shall, therefore, admit only spherically-symmetric initial surfaces $\tilde{\Sigma}$ and only that subgroup of $\text{Diff}\tilde{\mathcal{M}}$ (where $\tilde{\mathcal{M}} := \tilde{\Sigma} \times \mathbf{R}$), the elements of which commute with the rotations and are trivial at infinity. Let ρ , ϑ and φ be coordinates on $\tilde{\Sigma}$ that are adapted to the symmetry and $\rho \in [0, \infty)$, where $\rho = 0$ is the regular centre of symmetry; we assume that $\tilde{\Sigma}$ is smooth at this centre. The shell is at $\rho = \mathbf{r}$, and the infinity at $\rho = \infty$.

The coordinates ϑ and φ are ignorable coordinates; in the action, we can integrate over them so that they disappear and the effective initial manifold Σ is one-dimensional, diffeomorphic to \mathbf{R}_+ , and the effective background manifold \mathcal{M} is two-dimensional, $\mathbf{R}_+ \times \mathbf{R}$. Our restricted gauge group induces an effective gauge group, $\text{Diff}_{0,\infty}\mathcal{M}$, on \mathcal{M} ; it only contains diffeomorphisms that preserve the central boundary as well as, pointwise, the infinity.

Let us choose coordinates U and V on \mathcal{M} that satisfy the following boundary conditions at the gauge-invariant boundaries of \mathcal{M} : At the regular centre inside the shell,

$$U = V ,$$

at \mathcal{I}^- , $U = -\infty$ and $V \in (-\infty, \infty)$, at \mathcal{I}^+ , $V = \infty$ and $U \in (-\infty, \infty)$, and at i^0 , $U = -\infty$ and $V = \infty$. Otherwise, U and V are arbitrary.

Using these coordinates U and V , one can define the representative metric (see Sect. 2) by conditions on its components with respect to U and V . We shall choose them as follows.

1. U and V are double-null coordinates so that the representative line element takes the form

$$ds^2 = -A(U, V)dUdV + R^2(U, V)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \quad (5)$$

2. The representative metric is continuous at the shell.
3. For the outgoing shells, U is the retarded time determined by the representative metric at $V = \infty$. Analogously, for the in-going shells, V is the advanced time at $U = -\infty$.

Such a metric is uniquely defined for any physical situation given by the values of the parameters η , M , and w . This can be shown as follows.

Consider first the case $\eta = +1$. The Eddington-Finkelstein coordinate \tilde{U} satisfies already the conditions for U , so we need only to find the function V . In the Minkowski part, $U > u$, of the solution, the boundary conditions at the centre lead uniquely to:

$$A = 1 , \quad R = \frac{V - U}{2} . \quad (6)$$

In the Schwarzschild part, $U < u$, of the solution, V is an advanced null coordinate, so it must be some function, $V = X(M, u, \tilde{V})$, for each fixed M and u , of the advanced Eddington-Finkelstein coordinate \tilde{V} , which is defined by

$$\tilde{V} := U + 2R + 4M \ln \left| \frac{R}{2M} - 1 \right| .$$

The function X is uniquely determined by the boundary condition at the shell, requiring that V be continuous:

$$X(M, u, \tilde{V}|_{U=u}) = (U + 2R)_{U=u} ,$$

or,

$$X \left(M, u, u + 2R + 4M \ln \left| \frac{R}{2M} - 1 \right| \right) = u + 2R .$$

To solve this equation, we define

$$x := u + 2R + 4M \ln \left| \frac{R}{2M} - 1 \right| ,$$

calculate R in terms of M , u , and x , and substitute the result into the right-hand side:

$$X(M, u, x) = u + 2R(M, u, x). \quad (7)$$

A straightforward calculation yields

$$R(M, u, x) = 2M\kappa\left(\exp\left(\frac{x-u}{4M}\right)\right), \quad (8)$$

where κ is the well-known Kruskal function defined by its inverse,

$$\kappa^{-1}(y) = (y-1)e^y, \quad (9)$$

and $R > 2M$ was used. Eqs. (7) and (8) yield

$$V = u + 4M\kappa\left(\left(\frac{R}{2M} - 1\right)\exp\left(\frac{U-u+2R}{4M}\right)\right). \quad (10)$$

A similar calculation for $R < 2M$ leads to the same result. From this, it is easy to calculate R , if we observe that

$$\frac{R}{2M} + \ln\left(\frac{R}{2M} - 1\right) = \ln\left(\kappa^{-1}\left(\frac{R}{2M}\right)\right).$$

Then Eq. (9) implies that

$$R = 2M\kappa\left(\left(\frac{V-u}{4M} - 1\right)\exp\left(\frac{V-U}{4M}\right)\right). \quad (11)$$

This relation defines the desired transformation from $(U, R, \vartheta, \varphi)$ to $(U, V, \vartheta, \varphi)$.

As the last step, we calculate the metric for $U < u$ in the new coordinates. First, we differentiate the function R . The derivative of (9) determines the derivative of κ :

$$\kappa'(f) = \frac{1}{\kappa(f)e^{\kappa(f)}}, \quad (12)$$

which holds for any f . Then,

$$dR = \frac{1}{2\kappa(f_+)e^{\kappa(f_+)}} \left[-\left(\frac{V-u}{4M} - 1\right)\exp\left(\frac{V-U}{4M}\right)dU + \frac{V-u}{4M}\exp\left(\frac{V-U}{4M}\right)dV \right]$$

with

$$f_+ := \left(\frac{V-u}{4M} - 1\right)\exp\left(\frac{V-U}{4M}\right). \quad (13)$$

Now, (11) implies that

$$\frac{\kappa(f_+) - 1}{\kappa(f_+)} = 1 - \frac{2M}{R}, \quad (14)$$

and (9) that $f = (\kappa(f) - 1) \exp(\kappa(f))$. This leads to the relation

$$\frac{f_+}{\kappa(f_+)e^{\kappa(f_+)}} = 1 - \frac{2M}{R} ,$$

and thus

$$dR = -\frac{1}{2} \left(1 - \frac{2M}{R}\right) dU + \frac{1}{2\kappa(f_+)e^{\kappa(f_+)}} \frac{V-u}{4M} \exp\left(\frac{V-U}{4M}\right) dV .$$

Substituting this into the metric (2) results, finally, in

$$R = 2M\kappa(f_+), \quad A = \frac{1}{\kappa(f_+)e^{\kappa(f_+)}} \frac{V-u}{4M} \exp\left(\frac{V-U}{4M}\right), \quad (15)$$

where f_+ is defined by (13), cf. (11). With these expressions, it is easy to verify that A and R are continuous at the shell, as required. We note that these expressions contain u as well as M , which become conjugate variables in the canonical formalism. This makes the transition to the embedding variables non-trivial, and one must first look for this transformation on the constraint surface.

In the case of in-going shells ($\eta = -1$) a completely analogous procedure yields, for $V < v$, again (6), and for $V > v$,

$$R = 2M\kappa(f_-), \quad A = \frac{1}{\kappa(f_-)e^{\kappa(f_-)}} \frac{v-U}{4M} \exp\left(\frac{V-U}{4M}\right), \quad (16)$$

where

$$f_- := \left(\frac{v-U}{4M} - 1\right) \exp\left(\frac{V-U}{4M}\right) .$$

These expressions result from (15) by the substitution $V-u \rightarrow v-U$.

As the result of the gauge fixing, the set of solutions (η, M, w) can be written as a set of (η, M, w) -dependent metric fields (5) and a set of shell trajectories on a *fixed* background manifold \mathcal{M} . Here, the corresponding functions A and R have the form

$$A(\eta, M, w; U, V) , \quad R(\eta, M, w; U, V) , \quad (17)$$

and the trajectory of the shell on the background manifold is simply $U = u$ for $\eta = +1$ and $V = v$ for $\eta = -1$.

A key property of the background manifold is that it possesses a unique asymptotic region with \mathcal{I}^- defined by $U \rightarrow -\infty$ and \mathcal{I}^+ by $V \rightarrow +\infty$. As the shell cannot escape the background manifold, its reappearance at *an* asymptotic region must be interpreted as the reappearance at the asymptotic region of \mathcal{M} . In this way, the background manifold is a tool to solve the problem of where the shell reappears.

4 Transformation to embedding variables

4.1 Canonical formalism

The form of the canonical theory that is based on the embedding rather than ADM-type variables has been studied and advocated by Kuchař. In the recent paper [13] a large step forward in this field has been achieved. The embedding variables have been associated with background manifolds and gauge fixings similar to what has been done in the previous section. The existence of this transformation has been shown in the general case.

The resulting formalism inspires hopes that some unpleasant features of the ADM variables can be removed. First, the ADM variables lead to singular points in the physical configuration space (super-space [18, 19]) as well as at the constraint surface corresponding to spaces or space-times with symmetries. Second, the symmetry of the ADM theory itself is, on one hand, too large, containing all infinitesimal surface deformations, including also those transformations that do not result from diffeomorphisms. On the other hand, it is too small because only infinitesimal surface deformations and not finite group elements can act on the whole phase space. The constraint surface that has been constructed in [13] has, however, the form of a fibre bundle, which is a manifold (all points are regular), and the fibre group of this bundle is the diffeomorphism group of the background manifold, so it acts on the whole bundle.

As a Hamiltonian action principle that implies the dynamics of our system, we take the action Eq. (2.6) of [14] (see also [3]). Let us briefly summarise the relevant formulae. The spherically symmetric metric is written in the form:

$$ds^2 = -N^2 d\tau^2 + \Lambda^2 (d\rho + N^\rho d\tau)^2 + R^2 d\Omega^2 ,$$

and the shell is described by its radial coordinate $\rho = \mathbf{r}$. The action reads

$$S_0 = \int d\tau \left[\mathbf{p} \dot{\mathbf{r}} + \int d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R} - H_0) \right] , \quad (18)$$

and the Hamiltonian is

$$H_0 = N\mathcal{H} + N^\rho \mathcal{H}_\rho + N_\infty E_\infty ,$$

where $N_\infty := \lim_{\rho \rightarrow \infty} N^\rho(\rho)$, E_∞ is the ADM mass (see [14]), N and N^ρ are the lapse and shift functions, \mathcal{H} and \mathcal{H}_ρ are the constraints,

$$\mathcal{H} = \frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_\Lambda P_R}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} + \frac{\eta \mathbf{p}}{\Lambda} \delta(\rho - \mathbf{r}) , \quad (19)$$

$$\mathcal{H}_\rho = P_R R' - P'_\Lambda \Lambda - \mathbf{p} \delta(\rho - \mathbf{r}) , \quad (20)$$

and the prime (dot) denotes the derivative with respect to ρ (τ).

The main topic of this paper is to transform the variables in the action S_0 . This transformation will be split into two steps. The first step is a *transformation* of the canonical coordinates \mathbf{r} , \mathbf{p} , Λ , P_Λ , R , and P_R *at the constraint surface* Γ that is defined by the constraints (19) and (20). The new coordinates are u and $p_u = -M$ for $\eta = +1$, v and $p_v = -M$ for $\eta = -1$, and the so-called embedding variables $U(\rho)$ and $V(\rho)$.

The second step is an *extension* of the functions u , v , p_u , p_v , $U(\rho)$, $P_U(\rho)$, $V(\rho)$, and $P_V(\rho)$ *out of the constraint surface*, where the functions u , v , p_u , p_v , $U(\rho)$, and $V(\rho)$ are defined by the above transformation, and $P_U(\rho)$, $P_V(\rho)$ by $P_U(\rho)|_\Gamma = P_V(\rho)|_\Gamma = 0$. The extension must satisfy the condition that the functions form a canonical chart in a neighbourhood of Γ . A proof that such extension exists in general has been given in [13].

4.2 Transformation functions at the constraint surface

The constraint surface contains only points of the phase space that correspond to initial data for *solutions* of Einstein's equations. Hence, we can assume that the metric (5) is a spherically-symmetric solution with a shell, and so the functions $A(\eta, M, w; U, V)$ and $R(\eta, M, w; U, V)$ are those written down in the previous section, Eqs. (6), (15) and (16). According to Sect. 2, if such a metric is given, then, for each embedding, a unique first and second fundamental form can be calculated from it, and so the map from the embeddings to the ADM variables $q_{kl}(x)$ and $\pi^{kl}(x)$ can be constructed.

A very important point is to specify the family of embeddings that will be used throughout the paper. The embeddings are given by

$$U = U(\rho), \quad V = V(\rho) .$$

These functions have to satisfy several conditions.

1. As Σ is spacelike, U and V are null and increasing towards the future, we must have $U' < 0$ and $V' > 0$ everywhere.
2. At the regular centre, the four-metric is flat and the three-metric is to be smooth. This implies $U'(0) = -V'(0)$ in addition to the condition $U(0) = V(0)$. This follows from $T'(U(0), V(0)) = 0$ and means that Σ must run parallel to $T = \text{const.}$ in order to avoid conical singularities.
3. At infinity, the four-metric is the Schwarzschild metric. We require that the embedding approaches the Schwarzschild-time-constant surfaces $T = \text{const.}$, and that ρ becomes the Schwarzschild curvature coordinate R asymptotically.

More precisely, the behaviour of the Schwarzschild coordinates T and R along each embedding $U(\rho), V(\rho)$ must satisfy

$$T(\rho) = T_\infty + O(\rho^{-1}), \quad (21)$$

$$R(\rho) = \rho + O(\rho^{-1}). \quad (22)$$

The asymptotic coordinate T_∞ is a gauge-invariant quantity and it possesses the status of an observable.

4. At the shell ($\rho = \mathbf{r}$) we require the functions $U(\rho)$ and $V(\rho)$ to be C^∞ . In fact, as the four-metric is continuous in the coordinates U and V , but not smooth, only the C^1 -part of this condition is gauge invariant. Jumps in all higher derivatives are gauge dependent, but the condition will simplify equations.

We suppose further that there is a whole foliation of the solution space-times. Any foliation can be considered as a one-parameter family of embeddings:

$$U = U(\tau, \rho), \quad V = V(\tau, \rho),$$

the parameter being τ . The metric (5) reads, in terms of the coordinates τ, ρ, ϑ and φ :

$$ds^2 = -A\dot{U}\dot{V}d\tau^2 - A(\dot{U}V' + \dot{V}U')d\tau d\rho - AU'V'd\rho^2 + R^2d\Omega^2.$$

From this metric, we can read off the values of the variables Λ, R, N^ρ and N immediately:

$$\Lambda = \sqrt{-A(o, U, V)U'V'}, \quad R = R(o, U, V), \quad (23)$$

where o symbolises the observables (w and M , respectively), and

$$N = -\frac{\dot{U}V' - \dot{V}U'}{2U'V'}\sqrt{-AU'V'}, \quad N^\rho = \frac{\dot{U}V' + \dot{V}U'}{2U'V'}.$$

The expression $\dot{U}V' - \dot{V}U'$ is the Jacobian of the transformation from τ and ρ to U and V , and we assume it to be positive.

To calculate the gravitational momenta, we can use the canonical equations that follow when the action S_0 is varied with respect to P_Λ and P_R :

$$\begin{aligned} P_\Lambda &= -\frac{R}{N}(\dot{R} - N^\rho R'), \\ P_R &= -\frac{\Lambda}{N}(\dot{R} - N^\rho R') - \frac{R}{N}(\dot{\Lambda} - (N^\rho \Lambda)'). \end{aligned}$$

Substituting for R, Λ, N and N^ρ gives

$$\begin{aligned} \dot{\Lambda} - (N^\rho \Lambda)' &= \frac{1}{2\Lambda} \frac{\dot{U}V' - \dot{V}U'}{2U'V'} (-A_U U'^2 V' + A_V U' V'^2 - AU''V' + AU'V'') \\ \dot{R} - N^\rho R' &= \frac{\dot{U}V' - \dot{V}U'}{2U'V'} (R_U U' - R_V V'), \end{aligned}$$

so that, finally,

$$P_\Lambda = \frac{R}{\sqrt{-AU'V'}} (R_U U' - R_V V'), \quad (24)$$

$$P_R = R_U U' - R_V V' + \frac{RA_U}{2A} U' - \frac{RA_V}{2A} V' + \frac{R U''}{2 U'} - \frac{R V''}{2 V'}. \quad (25)$$

Here, the indices U and V denote the partial derivatives with respect to U and V .

Eqs. (23), (24), and (25) are the transformation equations expressing the variables Λ , R , P_Λ and P_R in terms of the new variables at the constraint surface. The functions A and R are given by (6), (15), and (16).

We now turn to the remaining variables η , \mathbf{r} and \mathbf{p} . We let η unchanged; in fact, we shall consider the action as two different actions, one for each value of η . The variable \mathbf{r} is related to our new variables u , v , M , $U(\rho)$ and $V(\rho)$ in a different way for each value of η . If $\eta = +1$, then \mathbf{r} is determined by the equation $U(\mathbf{r}) = u$. This is an equation with exactly one solution if u satisfies the condition $u < U(0)$ because $U(\rho)$ is a monotonous function with the range $(-\infty, U(0))$. For the differentials of the variables $U(\rho)$, \mathbf{r} and u , we obtain the relation:

$$d\mathbf{r} = \frac{du - dU(\mathbf{r})}{U'(\mathbf{r})}. \quad (26)$$

Similarly, if $\eta = -1$, then \mathbf{r} is defined by $V(\mathbf{r}) = v$ for $v > V(0)$, and the relation between the differentials takes the form:

$$d\mathbf{r} = \frac{dv - dV(\mathbf{r})}{V'(\mathbf{r})}. \quad (27)$$

The variable \mathbf{p} does not seem to be determined completely in [14] because equation (2.5a) of Ref. [14], which is the only equation that could serve this purpose, does not make sense in the limit $m \rightarrow 0$ of null shells. However, the constraint equations lead to some expressions for \mathbf{p} ; these determine \mathbf{p} only at the constraint surface, but this is, in fact, all we need. Let us, therefore, turn to the constraint equations.

4.3 The constraints

The constraint functions (19) and (20) contain finite parts, which are obtained for $\rho \neq \mathbf{r}$ and in the limits $\rho \rightarrow \mathbf{r} \pm$, and δ -function parts. The δ -function parts can be rewritten as equations for finite quantities, if one collects all terms with δ -function and sets the coefficient equal to zero.

From the boundary conditions at the shell and Eqs. (23), (24) and (25), it follows that the functions $\Lambda(\rho)$ and $R(\rho)$ are continuous, whereas $\Lambda'(\rho)$, $R'(\rho)$, $P_\Lambda(\rho)$ and

$P_R(\rho)$ jump across the shell, as the metric is not smooth. This implies in turn that the δ -function part of the constraints is equivalent to

$$\mathbf{p} = -\eta R[R'] , \quad (28)$$

$$\mathbf{p} = -\Lambda[P_\Lambda] , \quad (29)$$

where the symbol $[g] := g_+ - g_-$ denotes the jump of the quantity g across the shell.

Let us calculate the jumps. We have

$$[R'] = [R_U]U' + [R_V]V' .$$

For $\eta = +1$, we have to use (6) and (15) and to replace the limits $\rho \rightarrow \mathbf{r} \pm$ by $U \rightarrow u \pm$. We obtain immediately from (6) that

$$R_{U-} = -\frac{1}{2} , \quad R_{V-} = \frac{1}{2} .$$

Differentiating (15) with the help of formulae (12) and (13) leads to, for $U < u$,

$$R_{U+} = -\frac{f_+}{2\kappa(f_+)e^{\kappa(f_+)}} , \quad R_{V+} = \frac{\frac{V-u}{4M} \exp \frac{V-u}{4M}}{2\kappa(f_+)e^{\kappa(f_+)}} .$$

Eq. (6) and $\kappa(f_+) = R/2M$ imply for the limits that

$$\lim_{U \rightarrow u} \kappa(f_+) = \frac{V-u}{4M} ,$$

so we have

$$R_{U+} = -\frac{1}{2} + \frac{2M}{V-u} , \quad R_{V+} = \frac{1}{2} .$$

Hence,

$$[R_U] = \frac{2M}{V-u} , \quad [R_V] = 0 .$$

Similarly, for $\eta = -1$,

$$[R_U] = 0 , \quad [R_V] = -\frac{2M}{v-U} .$$

There is also the relation

$$R|_{U=u} = \frac{V-u}{2} , \quad R|_{V=v} = \frac{v-U}{2} ,$$

and so (28) yields:

$$\eta = +1 : \quad \mathbf{p} = -MU'(\mathbf{r}) , \quad (30)$$

$$\eta = -1 : \quad \mathbf{p} = -MV'(\mathbf{r}) . \quad (31)$$

For P_Λ , (24) implies

$$\Lambda[P_\Lambda] = R[R_U]U' - R[R_V]V' ,$$

and so (29) gives the same result as (28).

Let us return to the finite part of the constraints (19) and (20). If we substitute the above expressions for Λ , R , P_Λ , and P_R , we obtain, after some lengthy but straightforward calculation, for each $\rho \neq \mathbf{r}$:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2\Lambda}(4RR_{UV} + 4R_UR_V + A)U'V' + \\ & \frac{R}{A\Lambda}(AR_{UU} - A_UR_U)U'^2 + \frac{R}{A\Lambda}(AR_{VV} - A_VR_V)V'^2 , \end{aligned}$$

and

$$\mathcal{H}_\rho = -\frac{R}{A}(AR_{UU} - A_UR_U)U'^2 + \frac{R}{A}(AR_{VV} - A_VR_V)V'^2 .$$

If \mathcal{H} and \mathcal{H}_ρ are zero for any embedding outside the shell, that is for all possible U' and V' , the coefficients of $U'V'$, U'^2 and V'^2 must themselves vanish:

$$4RR_{UV} + 4R_UR_V + A = 0, \quad (32)$$

$$AR_{UU} - A_UR_U = 0, \quad (33)$$

$$AR_{VV} - A_VR_V = 0. \quad (34)$$

These three equations are equivalent to the full set of Einstein equations for any metric of the form (5). Thus, our functions A and R have to satisfy these equations. This is immediately clear for (6) which gives A and R inside the shell. A more tedious calculation verifies the validity of (32), (33), and (34) also outside the shell, where A and R are given by (15) and (16).

4.4 Transformation of the Liouville form

As it has been explained at the end of Sec. 4.1, the transformation of the action (18) to the new variables will be performed in two steps. The first step is restricted to the constraint surface and forms the content of the present section.

At the constraint surface, $H_0 = N_\infty E_\infty$ and the action (18) becomes

$$S_0|_\Gamma = \int d\tau \left[\mathbf{p}\dot{\mathbf{r}} - N_\infty E_\infty + \int d\rho (P_\Lambda \dot{\Lambda} + P_R \dot{R}) \right] .$$

According to the discussion given in [20], the ADM boundary term $N_\infty E_\infty$ in the action can, after parametrisation at the infinity, be written as $E_\infty \dot{T}_\infty$ and can be considered as a part of a modified Liouville form. Let us denote this form by Θ :

$$\Theta = \int d\rho (P_\Lambda d\Lambda + P_R dR) + \mathbf{p}d\mathbf{r} - E_\infty dT_\infty. \quad (35)$$

As a result, the transformation of the action is nothing but the transformation of the Liouville form Θ .

We expect that the terms remaining after the transformation do not depend on any embeddings, because the pull-back of the symplectic form to the constraint surface is degenerated exactly in the direction of the gauge variables $U(\rho)$ and $V(\rho)$. As we shall see, the constraint surface Γ consists of two components, Γ^+ and Γ^- , Γ^+ containing all outgoing and Γ^- all in-going shells. We split this form into three terms for the in-going and outgoing part, respectively,

$$\Theta|_{\Gamma^+} = \Theta^+|_{\Gamma^+} + \Theta^-|_{\Gamma^+} + \mathbf{p}d\mathbf{r},$$

where

$$\Theta^+|_{\Gamma^+} = \int_{\mathbf{r}}^{\infty} d\rho (P_{\Lambda}d\Lambda + P_RdR) - MdT_{\infty}$$

because $E_{\infty} = M$ at the constraint surface, and

$$\Theta^-|_{\Gamma^+} = \int_0^{\mathbf{r}} d\rho (P_{\Lambda}d\Lambda + P_RdR),$$

and similar expressions for $\Theta|_{\Gamma^-}$. Let us first transform the part $\Theta|_{\Gamma^+}$ of the Liouville form and make the ansatz

$$\Theta^+|_{\Gamma^+} \equiv \int_{\mathbf{r}}^{\infty} d\rho \vartheta - MdT_{\infty}, \quad (36)$$

with

$$\vartheta \equiv (fdU + gdV + h_ido^i)' + d\varphi, \quad (37)$$

where we have denoted the observables u and M collectively by o^i ($i = 1, 2$). This has to be compared with the corresponding part of (35), where the substitutions are made from (23),

$$\frac{d\Lambda}{\Lambda} = \frac{A_U}{2A}dU + \frac{A_V}{2A}dV + \frac{A_i}{2A}do^i + \frac{dU'}{2U'} + \frac{dV'}{2V'}, \quad (38)$$

$$dR = R_UdU + R_VdV + R_ido^i, \quad (39)$$

and P_{Λ} and P_R are given by (24) and (25).

It turns out to be convenient to make for the functions in (37) the ansatz

$$f = \frac{RR_U}{2} \ln \left(-\frac{U'}{V'} \right) + F(U, V, o^i), \quad (40)$$

$$g = \frac{RR_V}{2} \ln \left(-\frac{U'}{V'} \right) + G(U, V, o^i), \quad (41)$$

$$h_i = \frac{RR_i}{2} \ln \left(-\frac{U'}{V'} \right) + H_i(U, V, o^i), \quad (42)$$

$$\varphi = RR_UU' - RR_VV' - \frac{R}{2}(R_UU' + R_VV') \ln \left(-\frac{U'}{V'} \right) \quad (43)$$

$$-FU' - GV' + \phi(U, V, o^i). \quad (44)$$

The functions F, G, H_i, ϕ are then determined through comparison with the coefficients of dU, dV, dU', dV' , and $d\phi^i$. This leads to the equations

$$F_V - G_U = \frac{R}{2A}(2AR_{UV} - A_UR_V - A_VR_U) , \quad (45)$$

$$H_{iU} - F_i = -\frac{R}{2A}(2AR_{iU} - A_iR_U - A_UR_i) , \quad (46)$$

$$H_{iV} - G_i = \frac{R}{2A}(2AR_{iV} - A_iR_V - A_VR_i) , \quad (47)$$

$$\phi = 0 . \quad (48)$$

We next calculate the right-hand side of these equations by using the explicit expressions for A and R found in Sect. 2. Outside the shell, these are the expressions (15). It is convenient to introduce the abbreviations

$$b = \frac{V - u}{4M}, \quad a = \frac{U - u}{4M} . \quad (49)$$

One then has

$$A = \frac{be^{b-a}}{\kappa e^\kappa}, \quad R = 2M\kappa , \quad (50)$$

where κ is a function of $f_+ = (b-1)e^{b-a}$, see (13). The following identity turns out to be useful:

$$\frac{e^{b-a}}{e^\kappa} = \frac{\kappa - 1}{b - 1} . \quad (51)$$

After some lengthy, but straightforward calculations one finds

$$F_V - G_U = -\frac{\kappa - 1}{8b(b-1)} , \quad (52)$$

$$H_{uU} = F_u - \frac{\kappa - 1}{8b(b-1)} , \quad (53)$$

$$H_{uV} = G_u + \frac{\kappa - 1}{8b(b-1)} , \quad (54)$$

$$H_{MU} = F_M - \frac{1}{2} - \frac{\kappa - 1}{2(b-1)} , \quad (55)$$

$$H_{MV} = G_M + \frac{\kappa - 1}{2b(b-1)}a + \frac{\kappa^2 - b^2}{2b(b-1)} . \quad (56)$$

The freedom in the choice of solution to Eq. (45)–(47) enable us to set $F \equiv 0$. From the first equation one then gets

$$G = \int_u^U dU \frac{\kappa - 1}{8b(b-1)} = -\frac{M(\kappa^2 - b^2)}{4b(b-1)} , \quad (57)$$

where we have chosen the boundary condition that $G = 0$ for $U = u$, i.e., at the shell, and calculated the integral by the substitution $x = \kappa$. One recognises from

(57) that, at the shell, $G_M = 0$ and $G_u = -1/8b$. With this result for G , one can integrate Eqs. (53)–(56) for H_i and choose the integration constants such that

$$H_u = -G , \quad (58)$$

$$H_M = -\frac{1}{2}(U - u) - 4bG , \quad (59)$$

having $H_i = 0$, $i = 1, 2$, at the shell. This then yields for the Liouville form outside the shell pulled back to the constraint surface

$$\Theta^+|_{\Gamma^+} = (fdU + gdV + h_i do^i)|_{\mathbf{r}}^\infty + d\left(\int_{\mathbf{r}}^\infty d\rho \varphi\right) + d\mathbf{r}\varphi|_{\rho=\mathbf{r}} - MdT_\infty . \quad (60)$$

The fourth term on the right-hand side of (60) is a total derivative and will be omitted, since it does not contribute to the dynamics. Eqs. (26), (30) and (31) lead to

$$\mathbf{p}d\mathbf{r} = -M(du - dU) .$$

Analogously, one finds for the part $\Theta^-|_{\Gamma^+}$ inside the shell

$$\Theta^-|_{\Gamma^+} = (kdU + ldV)|_0^{\mathbf{r}} + d\left(\int_0^{\mathbf{r}} d\rho \psi\right) - d\mathbf{r}\psi|_{\rho=\mathbf{r}} , \quad (61)$$

with

$$k = \frac{RR_U}{2} \ln\left(-\frac{U'}{V'}\right) , \quad (62)$$

$$l = \frac{RR_V}{2} \ln\left(-\frac{U'}{V'}\right) , \quad (63)$$

$$\psi = RR_U U' - RR_V V' - \frac{R}{2}(R_U U' + R_V V') \ln\left(-\frac{U'}{V'}\right) . \quad (64)$$

Compared to f, g, h_i, φ , there are no terms analogous to G, F , and H_i , since the classical solutions (6) inside the shell lead to a vanishing right-hand side of (45)–(47). Because of the boundary condition $U'(0) = -V'(0)$, the functions k and l vanish at the centre. The third term on the right-hand side of (61) is again a total derivative and will be neglected.

One has therefore only potential contributions at the *shell* and at *infinity*. We shall consider first the contribution from the shell. Since there $F = G = H_u = H_M = 0$, one has to calculate

$$\begin{aligned} & (kdU + ldV)|_{\rho=\mathbf{r}} - d\mathbf{r}\psi|_{\rho=\mathbf{r}} - (fdU + gdV + h_u du - h_M dM)|_{\rho=\mathbf{r}} \\ & + d\mathbf{r}\varphi|_{\rho=\mathbf{r}} - M(du - dU) . \end{aligned} \quad (65)$$

Using (6) and (15), one arrives at the following jump conditions at the shell:

$$[RR_U] = M, [RR_V] = 0, [RR_u] = -M, [RR_M] = 0 . \quad (66)$$

Taking these into account, one recognises that *all* terms on the dust shell *cancel*. As we shall now demonstrate, the only non-vanishing terms are originating from *infinity*.

$$\Theta|_{\Gamma^+} = \lim_{\rho \rightarrow \infty} \left[H_i do^i + F dU + G dV + \ln \left(\frac{-U'}{V'} \right) \left(\frac{RR_U}{2} dU + \frac{RR_V}{2} dV + \frac{RR_i}{2} do^i \right) \right] - M dT_\infty, \quad (67)$$

where the function $F = 0$, and G , H_u and H_M are given by (57), (58), and (59), respectively. The limit (67) is determined by the boundary conditions 3 of Sec. 4.1, cf. (21) and (22).

Eqs. (21) and (22) determine the expansions of $U(\rho)$ and $V(\rho)$ uniquely. Indeed, for $\eta = +1$, U near the space-like infinity coincides with the Edington-Finkelstein retarded coordinate (see Sect. 3) and so is given in terms of T and R by

$$U = T - R - 2M \ln \left(\frac{R}{2M} - 1 \right).$$

Then,

$$U(\rho) = -\rho - 2M \ln \left(\frac{\rho}{2M} \right) + T_\infty + O(\rho^{-1}). \quad (68)$$

The presence of the logarithmic term is due to the long range of the gravitational potential. Thus, the first diverging term is universal, the second depends on the observable M , and the asymptotic coordinate T_∞ of the embedding appears only at the third position.

The asymptotic expansion of the function $V(\rho)$ can be determined from (10). We have first to get rid of κ :

$$(V - u - 4M) \exp \frac{V}{4M} = 2(R - 2M) \exp \left(\frac{U}{4M} + \frac{R}{2M} \right). \quad (69)$$

Then we substitute the expansions (22) and (68) into the right-hand side of (69):

$$(V - u - 4M) \exp \frac{V}{4M} = (\rho - 2M + O(\rho^{-1})) \exp \left[\frac{1}{4M} \left(\rho - 2M \ln \frac{\rho}{8M} + T_\infty \right) + O(\rho^{-1}) \right]. \quad (70)$$

Let us remove the singular part in the exponent by setting

$$V(\rho) = \rho - 2M \ln \frac{\rho}{8M} + T_\infty + V_1(\rho).$$

Eq. (70) then becomes

$$(1 - 2M\rho^{-1}) \exp O(\rho^{-1}) = \left[1 - 2M\rho^{-1} \ln \frac{\rho}{8M} + (T_\infty - u - 4M)\rho^{-1} + V_1(\rho)\rho^{-1} \right] \exp \frac{V_1(\rho)}{4M}. \quad (71)$$

Taking the limits $\rho \rightarrow \infty$ of both sides of (71), we obtain

$$\lim_{\rho \rightarrow \infty} \left(1 + \frac{V_1(\rho)}{\rho} \right) \exp \frac{V_1(\rho)}{4M} = 1.$$

It follows that

$$\lim_{\rho \rightarrow \infty} V_1(\rho) = 0 .$$

Eq. (71) implies then also that

$$V_1(\rho) = O\left(\frac{\ln \rho}{\rho}\right).$$

Hence, the expansion of the function $V(\rho)$ has the form:

$$V(\rho) = \rho - 2M \ln \frac{\rho}{8M} + T_\infty + O\left(\frac{\ln \rho}{\rho}\right); \quad (72)$$

the asymptotic coordinate T_∞ appears again only at the third position, and this is the reason why the expansion must be carried so far.

Now, the expansion of all functions contained in Θ is a straightforward matter. For G , we obtain from (57):

$$G = -\frac{M}{4} \frac{4R^2 - (V - u)^2}{(V - u)^2 - 4M(V - u)}.$$

Then, Eqs. (22), (68), and (72) give

$$G = -\frac{3}{4}M - 4M^2 \rho^{-1} \ln \frac{\rho}{8M} + M(2T_\infty - 2u - 3M)\rho^{-1} + o(\rho^{-1}) ,$$

where $o(\rho^{-1})$ is defined by the property $\lim_{\rho \rightarrow \infty} \rho o(\rho^{-1}) = 0$, and

$$H_u = -G = \frac{3}{4}M + 4M^2 \rho^{-1} \ln \frac{\rho}{8M} - M(2T_\infty - 2u - 3M)\rho^{-1} + O(\rho^{-2}) ,$$

as well as

$$H_M = \frac{5}{4}\rho + \frac{7}{2}M \ln \frac{\rho}{8M} + \frac{M}{2}(6 + 4 \ln 2) - \frac{7}{4}(T_\infty - u) + O(\rho^{-1}) .$$

Eq. (72) can be used to calculate dV :

$$dV = d\left(-2M \ln \frac{\rho}{8M}\right) + dT_\infty + O\left(\frac{\ln \rho}{\rho}\right) .$$

Then we obtain:

$$H_i do^i + F dU + G dV = M(dT_\infty - du) + dZ + O\left(\frac{\ln \rho}{\rho}\right) ,$$

where

$$Z = -\frac{7}{4}M(T_\infty - u) + \frac{5}{4}M\rho + \frac{5}{2}M^2 \ln \rho + \frac{1}{2}(4 - 13 \ln 2)M^2 - \frac{5}{2}M^2 \ln M.$$

Similarly,

$$\frac{R}{2} \ln \left(\frac{-U'}{V'} \right) = 2M + O(\rho^{-2}).$$

The derivatives R_U , R_V , R_u and R_M can be expanded if we calculate them from (11) using the identity (51):

$$R_U = -\frac{1}{2} \frac{R - 2M}{R}, \quad R_V = \frac{1}{2} \frac{R - 2M}{R} \frac{V - u}{V - u - 4M},$$

$$R_u = -2M \frac{R - 2M}{R} \frac{1}{V - u - 4M},$$

and

$$R_M = \frac{R}{M} - 2 \frac{R - 2M}{R} \left(\frac{1}{4M} \frac{V - U}{1 - \frac{4M}{V-u}} + \frac{U - u}{V - u - 4M} \right).$$

This gives:

$$\frac{R}{2} \ln \left(\frac{-U'}{V'} \right) (R_U dU + R_V dV + R_i d\sigma^i) = O\left(\frac{\ln \rho}{\rho}\right).$$

Collecting all terms, we finally have:

$$\Theta|_{\Gamma^+} = -Mdu + dZ + O\left(\frac{\ln \rho}{\rho}\right). \quad (73)$$

The exact form dZ can be omitted because it has no influence on the symplectic form and the equations of motion.

The final result of this subsection can be formulated as follows. The constraint surface Γ consists of two components: Γ^+ for the outgoing shells ($\eta = +1$), and Γ^- for the in-going shells ($\eta = -1$). On Γ^+ , we have the coordinates M , u , $U(\rho)$ and $V(\rho)$, and the pull-back of the Liouville form to Γ^+ is

$$\Theta|_{\Gamma^+} = -Mdu.$$

Thus, it is independent of $U(\rho)$ and $V(\rho)$, as expected.

In a completely analogous manner, the following result can be derived for the $\eta = -1$ case:

$$\Theta|_{\Gamma^-} = -Mdv,$$

and our coordinates on Γ^- are M , v , $U(\rho)$ and $V(\rho)$.

These results also show that the two Dirac observables $-M$ and u (or $-M$ and v) form a conjugate pair. Indeed, the Poisson algebra of Dirac observables is well-defined by the (degenerate) pull-back of the symplectic form to the constraint surface.

4.5 Extension to a neighbourhood of the constraint surface

In the previous subsection, the constraint surface pull-back of the Liouville form has been transformed to the Kuchař coordinates: the embeddings $U(\rho)$ and $V(\rho)$ that represent pure gauges, and p_u and u (or p_v and v) that are Dirac observables. The next task is to extend these coordinates to a neighbourhood of the constraint surface $\Gamma = \Gamma^+ \cup \Gamma^-$.

In [13], the proof has been given that such an extension exists if there are no points with additional symmetry at Γ . For the action (18), the space-time solutions are all spherically symmetric, but none of them exhibits any other symmetry, be it discrete or continuous. We can reformulate the result of [13] in a way suitable for our purposes as follows. There is a neighbourhood Γ'^\pm of Γ^\pm in the phase space, and functions

$$U, P_U, V, P_V, u, p_u, \quad (74)$$

in Γ'^+ and

$$U, P_U, V, P_V, v, p_v, \quad (75)$$

in Γ'^- such that, at Γ^\pm ,

$$P_U(\rho) = P_V(\rho) = 0,$$

and $U(\rho)$, $V(\rho)$, p_u and u (or p_v and v) coincide with our coordinates there. The functions (74) and (75) form canonical charts in Γ'^\pm . The transformation between the old variables

$$R, P_R, \Lambda, P_\Lambda, \mathbf{r}, \mathbf{p}, \quad (76)$$

and the new ones Eq. (74) or (75) is smooth and invertible in Γ'^\pm . At the constraint surface Γ^\pm , the transformation is given by Eqs. (23)–(25), (15), (16), $U(\mathbf{r}) = u$, $V(\mathbf{r}) = v$, (30) and (31). Outside the constraint surface, only the existence of the transformation has been shown so we do not know its form.

Using this result, we can write the transformed action S^\pm in Γ'^\pm as follows,

$$S^+ = \int_{-\infty}^{\infty} d\tau \left[p_u \dot{u} + \int_0^{\infty} d\rho \left(P_U(\rho) \dot{U}(\rho) + P_V(\rho) \dot{V}(\rho) - N_U(\rho) P_U(\rho) - N_V(\rho) P_V(\rho) \right) \right],$$

and

$$S^- = \int_{-\infty}^{\infty} d\tau \left[p_v \dot{v} + \int_0^{\infty} d\rho \left(P_U(\rho) \dot{U}(\rho) + P_V(\rho) \dot{V}(\rho) - N_U(\rho) P_U(\rho) - N_V(\rho) P_V(\rho) \right) \right].$$

The two actions can be considered as the reduced form of just one action. Consider the case $\eta = +1$ first. The dynamical trajectory of the shell is given by

the relation $u(\tau) = \text{const}$, whereas $v(\tau)$ is arbitrary, depending on the choice of the parameter τ (the only restriction is that $v(\tau)$ is an increasing function). This information can be obtained from the extended action:

$$S_{\text{ext}}^+ = \int_{-\infty}^{\infty} d\tau \left[p_u \dot{u} + p_v \dot{v} - n_u p_v + \int_0^{\infty} d\rho \left(P_U(\rho) \dot{U}(\rho) + P_V(\rho) \dot{V}(\rho) - N_U(\rho) P_U(\rho) - N_V(\rho) P_V(\rho) \right) \right] ,$$

where n_u is a Lagrange multiplier, $v(\tau)$ a pure gauge and p_v -dependent. Similarly for $\eta = -1$:

$$S_{\text{ext}}^- = \int_{-\infty}^{\infty} d\tau \left[p_u \dot{u} + p_v \dot{v} - n_v p_u + \int_0^{\infty} d\rho \left(P_U(\rho) \dot{U}(\rho) + P_V(\rho) \dot{V}(\rho) - N_U(\rho) P_U(\rho) - N_V(\rho) P_V(\rho) \right) \right] .$$

One can set in S_{ext}^+ , as $p_u = -M < 0$:

$$n_u = n p_u$$

and, similarly, in S_{ext}^- ,

$$n_v = n p_v .$$

Then, clearly, S_{ext}^+ and S_{ext}^- are obtained by reducing the following action

$$S = \int_{-\infty}^{\infty} d\tau \left[p_u \dot{u} + p_v \dot{v} - n p_u p_v + \int_0^{\infty} d\rho \left(P_U(\rho) \dot{U}(\rho) + P_V(\rho) \dot{V}(\rho) - N_U(\rho) P_U(\rho) - N_V(\rho) P_V(\rho) \right) \right] .$$

Indeed, the case $\eta = +1$ ($\eta = -1$) is obtained from the solution $p_v = 0$ ($p_u = 0$) of the constraint

$$p_u p_v = 0 .$$

The relation between the total energy M and the two momenta p_u and p_v can now be written as follows:

$$M = -p_u - p_v .$$

5 Conclusions

We have demonstrated that there is a transformation of variables bringing the action (18) to the simple form of the so-called *Kuchař decomposition*

$$S = \int d\tau (p_u \dot{u} + p_v \dot{v} - n p_u p_v) + \int d\tau \int_0^{\infty} d\rho (P_U \dot{U} + P_V \dot{V} - H) , \quad (77)$$

where $H = N^U P_U + N^V P_V$; n , $N^U(\rho)$ and $N^V(\rho)$ are the new Lagrange multipliers.

The dependence of the new variables P_U and P_V on the old ones is not known. This dependence would be needed for calculation of the spacetime geometry associated with any solution given in terms of the new variables. We know, however, that the new constraint equations, $P_U(\rho) = P_V(\rho) = 0$, are mathematically equivalent to the old constraints, Eqs. (19) and (20). One can, therefore, use the old constraints to calculate the geometry from the true degrees of freedom along the hypersurfaces of some foliation. The fact that two spacetimes obtained by this method using different foliations are isometric is guaranteed by the closure of the algebra of constraints [21].

The new phase space has non-trivial boundaries:

$$p_u \leq 0, \quad p_v \leq 0, \quad (78)$$

$$\frac{-u + v}{2} > 0, \quad (79)$$

$$p_v = 0, \quad U \in (-\infty, u), \quad V > u - 4p_u \kappa \left(-\exp \frac{u - U}{4p_u} \right), \quad (80)$$

and

$$p_u = 0, \quad V \in (v, \infty), \quad U < v + 4p_v \kappa \left(-\exp \frac{V - v}{4p_v} \right). \quad (81)$$

The boundaries defined by inequalities (79)–(81) are due to the singularity.

The two dynamical systems defined by the actions (18) and (77) are equivalent: each maximal dynamical trajectory of the first, if transformed to the new variables, give a maximal dynamical trajectory of the second and vice versa.

The variables u , v , p_u and p_v span the effective phase space of the shell. They contain all true degrees of freedom of the system. One can observe that the corresponding part of the action (77) coincides with the action for free motion of a zero-rest-mass spherically-symmetric (light) shell in flat spacetime if one replaces the inequality (79) by

$$\frac{-u + v}{2} \geq 0.$$

Such a dynamics is complete if the singularity at the value zero of the radius of the shell, $(-u + v)/2$, can be considered as a harmless caustic so that the light can re-expand after passing through it. It might, therefore, seem also possible to extend the phase space of the gravitating shell in the same way so that the in-going and the out-going sectors are merged together into one bouncing solution.

However, such a formal extension of the dynamics (77) is not adequate. The physical meaning of any solution written in terms of new variables (74) or (75) is given by measurable quantities of geometrical or physical nature such as the curvature of spacetime or the density of matter. These observables must be expressed as functions on the phase space. They can of course be transformed between the

phase spaces of the two systems (18) and (77). They cannot be left out from any complete description of a system, though they are often included only tacitly: an action alone does not define a system. This holds just as well for the action (18) as for (77).

Let us consider these observables. The expression for the stress-energy tensor of the shell written down in [14] implies that the density of matter diverges at $\mathbf{r} = 0$; this corresponds to $(-u + v)/2 = 0$ in terms of the new variables. Eqs. (15) and (16) can be used to show that the curvature of the solution spacetime diverges at the boundary defined by Eq. (80) for $p_v = 0$ and by Eq. (81) for $p_u = 0$. It follows that the observable quantities at and near the “caustic” are badly singular and that there is no sensible extension of the dynamics defined by action (77) to it, let alone through it. This confirms the more or less obvious fact that no measurable property (such as the singularity) can be changed by a transformation of variables.

The action for the null dust shell is now written in a form which can be taken as the starting point for quantisation. Surprisingly, it will turn out that the quantum theory is, in fact, singularity-free. This will be done in a separate paper [22].

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